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Approximation of the Lévy–Feller advection–dispersion process by random walk and finite difference method

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Abstract

In this paper we present a random walk model for approximating a Lévy–Feller advection–dispersion process, governed by the Lévy–Feller advection–dispersion differential equation (LFADE). We show that the random walk model converges to LFADE by use of a properly scaled transition to vanishing space and time steps. We propose an explicit finite difference approximation (EFDA) for LFADE, resulting from the Grünwald–Letnikov discretization of fractional derivatives. As a result of the interpretation of the random walk model, the stability and convergence of EFDA for LFADE in a bounded domain are discussed. Finally, some numerical examples are presented to show the application of the present technique.

Keywords: Lévy–Feller advection–dispersion process; Finite difference approximation; Discrete random walk model; Stability analysis; Convergence analysis

1. Introduction

Recently a growing number of researchers have utilized fractional calculus in a variety of applied fields resulting in fractional differential equations being used across many fields of science and engineering [1–4]. Liu et al. [5–7] simulated Lévy motion with α-stable densities using a fractional advection–dispersion equation. Lynch [8] discussed a possible mechanism underlying plasma transport in magnetically confined plasmas. Gorenflo and Mainardi [9], Gerenflo and Vivoli [10] presented a probability density function for diffusion limits. Diethelm [11] described physical phenomena such as damping laws and diffusion processes via fractional differential equations. As is well-known, analytic solutions of most fractional differential equations cannot be obtained explicitly, so many authors resort to numerical solution strategies based on convergence and stability analyses [8,12–14].

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Gorenflo and Abdel-Rehim [15] and Abdel-Rehim [16] proposed discrete approximations to spatially one-dimensional time-fractional diffusion processes with drift towards the origin, by generalization of Ehrenfest's urn model. Then they interpreted discrete approximations (a) as difference schemes (explicit and implicit), (b) as random walk models, and discussed their convergence from the probabilistic standpoint, instead of strong convergence in the supremum norm discussed in the present paper.

One type of fractional differential equations, the fractional advection–dispersion equation, is used in groundwater hydrology research to model the transport of passive tracers carried by fluid flow in a porous medium. Meerschaert and Tadjeran [17] presented numerical methods to solve the one-dimensional fractional advection–dispersion equations with variable coefficients on a finite domain.

Recently some authors discussed the Lévy–Feller diffusion process, and demonstrated that it could be dealt with by a generalized diffusion equation [19–21]. We stress that Gorenflo and Mainardi [19,20] have coined the name "Lévy–Feller diffusion process", and they presented a random walk model for approximating the Lévy–Feller diffusion process and produced sample paths of individual particles performing the random walk using Monte Carlo simulation. They proved weak convergence (also called "convergence in distribution" or "convergence in law") of the discrete solution towards the probability law of the process.

In this paper, a drift term is added to the Lévy–Feller diffusion equation. Following Gorenflo and Mainardi [19,20] and Gorenflo et al. [21], we call the described process a "Lévy–Feller advection–dispersion process". In contrast to [19–21], we have extended the processes to the case of bounded spatial domain and for this situation we give an analysis of stability and convergence in the supremum norm which is appropriate in numerical analysis.

We first introduce some notations of the Lévy–Feller diffusion process adopted in [19–21] and use this notation throughout the paper. Feller [22] investigated the semigroups of one-dimensional pseudo-differential operators arising by inversion of linear combinations of left and right hand sided Riemann–Liouville operators. These semigroups describe space-fractional diffusion processes evolving in time. Lévy [23] interpreted the semigroups as stable distributions of some stochastic processes from the probabilistic standpoint.

The Lévy–Feller diffusion was then introduced for studying stable stochastic Markovian processes. Let $p_\alpha(x;\theta)$ denote for $x \in \mathbb{R}$, $|\theta| \leqslant 2-\alpha$, $1<\alpha\leqslant 2$ the stable probability distribution whose characteristic function (Fourier transform) [20] is

$$\hat{p}_\alpha(k;\theta) = \exp(-|k|^\alpha e^{\mathrm{isign}(k)\theta\pi/2}), \quad (k \in \mathbb{R}). \tag{1}$$

Introducing the similarity variable $xt^{-\frac{1}{\alpha}}$, we obtain

$$g_\alpha(x,t;\theta) = t^{-\frac{1}{\alpha}} p_\alpha\left(xt^{-\frac{1}{\alpha}};\theta\right), \quad (x\in\mathbb{R}, t>0), \tag{2}$$

where x is the space variable and t the time variable. The Fourier transform of $g_\alpha(x,t;\theta)$ is

$$\hat{g}_\alpha(\kappa,t;\theta) = \exp(-t|\kappa|^\alpha e^{\mathrm{isign}(k)\theta\pi/2}), \quad (\kappa\in\mathbb{R}). \tag{3}$$

The function $g_\alpha(x,t;\theta)$ is considered as the fundamental solution of the generalized diffusion equation

$$\frac{\partial u(x,t)}{\partial t} = D_\theta^\alpha u(x,t), \quad (x\in\mathbb{R}, t>0), \tag{4}$$

where the operator D_θ^α is the Riesz–Feller fractional derivative (in space) of order α and skewness θ.

In this paper, we discuss the Lévy–Feller advection–dispersion equation (LFADE) including an advection term:

$$\frac{\partial u(x,t)}{\partial t} = aD_\theta^\alpha u(x,t) - b\frac{\partial u(x,t)}{\partial x} \tag{5}$$

with initial condition:

$$u(x,0) = \varphi(x), \tag{6}$$

where $x\in\mathbb{R}$, $t>0$, and $a>0$, $b\geqslant 0$.

The fundamental solution of (5) and (6) has been derived using the Fourier transform [24] as

$$\widehat{G}_\alpha(\kappa, t; \theta) = \exp(-ta|\kappa|^\alpha e^{\mathrm{i}\,\mathrm{sign}(k)\theta\pi/2} + \mathrm{i}tb\kappa), \quad (\kappa \in \mathbb{R}). \tag{7}$$

As mentioned above, Feller [22] has showed that the pseudo-differential operator D_θ^α can be viewed as the inverse to the Feller potential, which is a linear combination of two Riemann–Liouville differential operators.

We introduce the Riemann–Liouville integral:

$$\begin{cases} I_+^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_{-\infty}^x (x-\xi)^{\alpha-1} f(\xi)\,\mathrm{d}\xi, \\ I_-^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_x^{+\infty} (\xi - x)^{\alpha-1} f(\xi)\,\mathrm{d}\xi \end{cases} \tag{8}$$

and the coefficients

$$\begin{cases} c_+ = c_+(\alpha, \theta) := \frac{\sin((\alpha-\theta)\pi/2)}{\sin(\alpha\pi)}, \\ c_- = c_-(\alpha, \theta) := \frac{\sin((\alpha+\theta)\pi/2)}{\sin(\alpha\pi)}. \end{cases} \tag{9}$$

It is easily proved that $c_\pm \leqslant 0$ when $1 < \alpha \leqslant 2$.

Following the notation by Gorenflo and Mainardi [19], the Feller potential reads

$$I_\theta^\alpha f(x) = c_+(\alpha, \theta) I_+^\alpha f(x) + c_-(\alpha, \theta) I_-^\alpha f(x).$$

In [22] Feller showed that the operator I_θ^α possesses the semigroup property

$$I_\theta^\alpha I_\theta^\beta = I_\theta^{\alpha+\beta} \quad \text{for} \quad 0 < \alpha, \beta < 1 \text{ and } \alpha + \beta < 1.$$

Using the Feller potential, Gorenflo et al. [19] defined the Riesz–Feller operator

$$D_\theta^\alpha := -I_\theta^{-\alpha} = -[c_+(\alpha, \theta) I_+^{-\alpha} + c_-(\alpha, \theta) I_-^{-\alpha}], \quad (1 < \alpha \leqslant 2), \tag{10}$$

where $I_+^{-\alpha}$ and $I_-^{-\alpha}$ are the inverse of the integral operators I_+^α and I_-^α respectively. For integral representations of the Riemann–Liouville fractional derivative operators $I_\pm^{-\alpha}$ [2], we have

$$I_\pm^{-\alpha} = \frac{\mathrm{d}^2}{\mathrm{d}x^2} I_\pm^{2-\alpha}.$$

In particular, we have $D_0^2 = \frac{\mathrm{d}^2}{\mathrm{d}x^2}$.

The introduction of Feller's and Riemann–Liouville's considerations helps us construct a difference scheme via the Grünwald–Letnikov discretization of fractional derivatives, which is interpreted as a discrete random walk model. We then prove that the discrete random walk model converges to the Lévy–Feller advection–dispersion process.

This paper is organized as follows. In Section 2, we discuss the discrete random walk approach to the LFADE, which is based on the well-known Grünwald–Letnikov discretization of fractional derivatives. In Section 3, we discuss the convergence and domain of attraction. We prove that the discrete probability distribution generated by the random walk model belongs to the domain of attraction of the corresponding stable distribution. In Section 4 we propose an explicit finite difference scheme for solving LFADE. In Section 5 we give the stability and convergence analyses of the numerical scheme. Finally, numerical results are presented to show the application of the present technique.

2. Discrete random walk in space and time

In this section, we present a discrete random walk model for the LFADE with the initial condition:

$$u(x, 0) = \delta(x), \quad (x \in \mathbb{R}), \tag{11}$$

where $\delta(x)$ is the Dirac delta function.

We discretize space and time by the grid points

$$x_j = jh, \quad h > 0, \;\; j = 0, \pm 1, \pm 2, \dots$$

and time instants

$$t_n = n\tau, \quad \tau > 0, \;\; n = 0, 1, 2, \dots.$$

The dependent variable u is then discretized by introducing $y_j(t_n)$ as

$$y_j(t_n) = \int_{x_j - h/2}^{x_j + h/2} u(x, t_n)\, \mathrm{d}x \approx hu(x_j, t_n).$$

To obtain a random walk model for LFADE, we approximate the first-order derivatives $\frac{\partial u}{\partial t}$ and $\frac{\partial u}{\partial x}$ in LFADE by using the first-order quotient. We assume that the solution has suitable properties, (i.e., it has first-order continuous derivatives and its second-order derivative is integrable) so that the function's α-order derivatives in both Riemann–Liouville and Grünwald–Letnikov senses are the same. According to this property we discretize the operator D_θ^α in LFDAE using the definition of Grünwald–Letnikov fractional derivative:

$$I_\pm^{-\alpha} = \lim_{h \to 0} {}_hI_\pm^{-\alpha},$$

where ${}_hI_\pm^{-\alpha}$ denote the approximation for the shifted Grünwald–Letnikov operators, which read

$${}_hI_\pm^{-\alpha} f(x) = \frac{1}{h^\alpha} \sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} f(x \mp (k-1)h). \tag{12}$$

Discretizing all variables, we replace LFADE by the finite difference equation

$$\frac{y_j(t_{n+1}) - y_j(t_n)}{\tau} = a_h D_\theta^\alpha y_j(t_n) - b\frac{y_j(t_n) - y_{j-1}(t_n)}{h}, \tag{13}$$

where the difference operator ${}_hD_\theta^\alpha$ reads

$${}_hD_\theta^\alpha y_j(t_n) = -[c_{+h} I_+^{-\alpha} y_j(t_n) + c_{-h} I_-^{-\alpha} y_j(t_n)]. \tag{14}$$

In view of the operator (14), the operators ${}_hI_\pm^{-\alpha}$ (12) are given by

$${}_hI_\pm^{-\alpha} y_j(t_n) = \frac{1}{h^\alpha} \sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} y_{j\pm 1 \mp k}(t_n). \tag{15}$$

We now introduce the concept of a discrete random walk model. A discrete random walk on the grids $(jh|j \in Z)$ is obtained by defining the random variables:

$$S_n = hY_1 + hY_2 + \cdots + hY_n, \quad (n \in N),$$

where $S_0 = 0$; $Y_1, Y_2, \dots, Y_n$ are independent identically distributed random variables. Discretizing the space variable x and the time variable t, the recursion $S_{n+1} = S_n + hY_{n+1}$ (following from the above definition of random variables) implies that

$$y_j(t_{n+1}) = \sum_{k=-\infty}^{+\infty} p_k y_{j-k}(t_n), \quad (j \in Z, n \in N_0). \tag{16}$$

By a suitable normalization, the $y_j(t_n)$ may be interpreted as the probability of sojourn in point x_j at time t_n for a particle making a discrete random walk on the spatial grids in discrete instants. When time proceeds from $t = t_n$ to $t = t_{n+1}$, the sojourn probabilities are redistributed according to the general rule (16). p_k denotes a suitable transfer coefficient, which represents the probability of transition from x_{j-k} to x_j (likewise from x_j to x_{j-k}) and is spatially homogeneous and time stationary; $y_j(0)$ denotes the probability of sojourn of the random walker in point x_j at instant $t_0 = 0$. Using the definition and the property of the Dirac delta function $\delta(x)$, we have

$$y_j(0) = \int_{x_j - h/2}^{x_j + h/2} u(x, 0)\, \mathrm{d}x = \int_{x_j - h/2}^{x_j + h/2} \delta(x)\, \mathrm{d}x = \begin{cases} 1, & j = 0, \\ 0, & j \neq 0. \end{cases} \tag{17}$$

It is clear that this means that the random walker starts at point $x_0 = 0$. Actually, the formula (16) can be interpreted as a discrete random model only if p_k satisfies

$$\sum_{k=-\infty}^{+\infty} p_k = 1, \quad p_k \geqslant 0, \;\; k = 0, \pm 1, \pm 2, \ldots \tag{18}$$

Using (14) and (15), the finite difference Eq. (13) becomes

$$y_j(t_{n+1}) = y_j(t_n) - \frac{a\tau}{h^\alpha} \sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} [c_+ y_{j+1-k}(t_n) + c_- y_{j-1+k}(t_n)] - \frac{b\tau}{h}[y_j(t_n) - y_{j-1}(t_n)]. \tag{19}$$

The transition coefficients in (16) are easily deduced from (19) and are given by

$$\begin{cases} p_0 = 1 + \frac{a\tau}{h^\alpha}\binom{\alpha}{1}(c_+ + c_-) - \frac{b\tau}{h}, \\ p_{+1} = -\frac{a\tau}{h^\alpha}\left(c_+\binom{\alpha}{2} + c_-\right) + \frac{b\tau}{h}, \\ p_{-1} = -\frac{a\tau}{h^\alpha}\left(c_+ + c_-\binom{\alpha}{2}\right), \\ p_{\pm k} = (-1)^k \frac{a\tau}{h^\alpha} c_\pm \binom{\alpha}{k+1}, \quad k = 2, 3, \ldots \end{cases} \tag{20}$$

To interpret the difference scheme (19) as a discrete random walk model, we have to check whether the coefficients (20) satisfy the conditions (18). Directly from (19) it can be easily verified that

$$\sum_{k=-\infty}^{+\infty} p_k = 1 - \frac{a\tau}{h^\alpha} \sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} (c_+ + c_-) - \frac{b\tau}{h}(1-1) = 1 - \frac{a\tau}{h^\alpha}(c_+ + c_-)(1-1)^\alpha = 1.$$

Observe that all $p_{\pm k} \geqslant 0$, $k \in N$ because of the inequalities$(-1)^k \binom{\alpha}{k+1} < 0$ and $c_\pm \leqslant 0$ for $1 < \alpha \leqslant 2$, whereas $p_0 \geqslant 0$ under the condition

$$0 \leqslant 1 + \frac{a\tau}{h^\alpha}\alpha(c_+ + c_-) - \frac{b\tau}{h} < 1.$$

Therefore, the time step τ and the space step h are subject to the constraint

$$-\frac{a\tau}{h^\alpha}\alpha(c_+ + c_-) + \frac{b\tau}{h} \leqslant 1 \tag{21}$$

or a sufficient condition of the scaling constraint:

$$\mu = \frac{\tau}{h^\alpha} \leqslant \frac{1}{-a\alpha(c_+ + c_-) + b}. \tag{22}$$

Clearly if the condition (22) of the scaling constraint is satisfied and consequently the constraint (21) is also satisfied.

3. Convergence of the random walk model to a stable probability distribution

In this section, using the notations and techniques in [20], we will prove that the random walk model in the above section converges completely to a stable probability distribution. The probability distribution has the characteristic function (7).

Let us consider the generation functions

$$\tilde{p}(z) = \sum_{j=-\infty}^{+\infty} p_j z^j, \quad \tilde{y}_n(z) = \sum_{j=-\infty}^{+\infty} y_j(t_n) z^j, \quad |z| = 1 \tag{23}$$

for the transition probabilities (transfer coefficients) p_k and the sojourn probabilities $y_j(t_n)$, respectively. From the property of $y_j(0)$ in (17), we obtain

$$\tilde{y}_0(z) = \sum_{j=-\infty}^{+\infty} y_j(0)z^j = y_0(0)z^0 = 1.$$

From the discrete convolution (16), we have

$$\tilde{y}_n(z) = \tilde{y}_0(z)[\tilde{p}(z)]^n = [\tilde{p}(z)]^n, \quad (n \in N).$$

The two power series in (23) are absolutely and uniformly convergent. Putting $z = \mathrm{e}^{\mathrm{i}kh}$, $k \in \mathbb{R}$, we obtain $\tilde{p}(z) = \tilde{p}(\mathrm{e}^{\mathrm{i}kh})$ and $\tilde{y}(z, t_n) = \tilde{y}(\mathrm{e}^{\mathrm{i}kh}, t_n)$.

When fixing the parameter μ as a positive number subject to the restriction (22), and letting the space step h (and likewise τ) go to zero, we have $n = \frac{t}{\tau} = \frac{t}{\mu h^\alpha} \to \infty$. Letting $t = t_n$, we obtain

$$\tilde{y}(z, t) = [\tilde{p}(z)]^{t/\tau}.$$

Putting $z = \mathrm{e}^{\mathrm{i}kh}$, the above formula gives

$$\tilde{y}(\mathrm{e}^{\mathrm{i}kh}, t) = [\tilde{p}(\mathrm{e}^{\mathrm{i}kh})]^{t/\tau}. \tag{24}$$

In the following we present a result which can be used to deduce the convergence of the random walk model sequentially. From $z = \mathrm{e}^{\mathrm{i}kh}$ and (23), $\tilde{y}(\mathrm{e}^{\mathrm{i}kh}, t)$ can be viewed as the discrete Fourier transform for the numerical solution of LFADE. It can be seen that if $\tilde{y}(\mathrm{e}^{\mathrm{i}kh}, t)$ tends to $\widehat{G}_\alpha(\kappa, t; \theta)$, when the space step h tends to zero, the random walk model (19) can be viewed to approximate the LFADE.

Thus, we have to prove the following result:

Theorem 1. *If it suffices that, for fixed k,*

$$\tilde{y}(\mathrm{e}^{\mathrm{i}kh}, t) \to \exp(-t|k|^\alpha \mathrm{e}^{\mathrm{i}\,\mathrm{sign}(k)\theta\pi/2} + \mathrm{i}tbk), \quad as \quad h \to 0, \tag{25}$$

the difference scheme (19) interpreted as a discrete random walk can be viewed to approximate the related equation LFADE.

Proof. With the definition of $\tilde{p}(z)$ in (23), the coefficients in (20), and using the binomial series for $(1-z)^\alpha$, the following equations are obtained:

$$\begin{aligned}
\tilde{p}(z) &= p_0 + \sum_{k=1}^{\infty}(p_k z^k + p_{-k}z^{-k}) \\
&= 1 + \frac{a\tau}{h^\alpha}\binom{\alpha}{1}(c_+ + c_-) - \frac{b\tau}{h} - \frac{a\tau}{h^\alpha}\left(c_+\binom{\alpha}{2} + c_-\right)z + \frac{b\tau}{h}z \\
&\quad - \frac{a\tau}{h^\alpha}\left(c_+ + c_-\binom{\alpha}{2}\right)z^{-1} + \sum_{k=2}^{\infty}(-1)^k\frac{a\tau}{h^\alpha}\binom{\alpha}{k+1}(c_+z^k + c_-z^{-k}) \\
&= 1 - \frac{a\tau}{h^\alpha}\left[c_+\left(-\binom{\alpha}{1} + \binom{\alpha}{2}z + \binom{\alpha}{0}z^{-1} + z^{-1}\sum_{k=3}^{\infty}(-1)^k\binom{\alpha}{k}z^k\right)\right. \\
&\quad \left. + c_-\left(-\binom{\alpha}{1} + \binom{\alpha}{2}z^{-1} + \binom{\alpha}{0}z + z\sum_{k=3}^{\infty}(-1)^k\binom{\alpha}{k}z^{-k}\right)\right] - \frac{b\tau}{h}(1-z) \\
&= 1 - \frac{a\tau}{h^\alpha}[c_+z^{-1}(1-z)^\alpha + c_-z(1-z^{-1})^\alpha] - \frac{b\tau}{h}(1-z).
\end{aligned}$$

Putting $z = \mathrm{e}^{\mathrm{i}kh}$, the above formula is rewritten as

$$\tilde{p}(\mathrm{e}^{\mathrm{i}kh}) = 1 - \frac{a\tau}{h^\alpha}[c_+\mathrm{e}^{-\mathrm{i}kh}(1-\mathrm{e}^{\mathrm{i}kh})^\alpha + c_-\mathrm{e}^{\mathrm{i}kh}(1-\mathrm{e}^{-\mathrm{i}kh})^\alpha] - \frac{b\tau}{h}(1-\mathrm{e}^{\mathrm{i}kh}).$$

For small h, Taylor's theorem gives

$$1-\mathrm{e}^{\pm \mathrm{i}kh} = \mp \mathrm{i}kh + \mathrm{O}(h^2),$$
$$\mathrm{e}^{\pm \mathrm{i}kh} = 1 + \mathrm{O}(h).$$

Hence,

$$\mathrm{O}(h)(1-\mathrm{e}^{\pm \mathrm{i}kh})^\alpha = \mathrm{O}(h^{\alpha+1}).$$

We then obtain

$$\begin{aligned}\tilde{p}(\mathrm{e}^{\mathrm{i}kh}) &= 1 - \frac{a\tau}{h^\alpha}[c_+(1+\mathrm{O}(h))(1-\mathrm{e}^{\mathrm{i}kh})^\alpha + c_-(1+\mathrm{O}(h))(1-\mathrm{e}^{-\mathrm{i}kh})^\alpha + \mathrm{O}(h^{\alpha+1})] - \frac{b\tau}{h}(-\mathrm{i}kh + \mathrm{O}(h^2))\\ &= 1 - \frac{a\tau}{h^\alpha}[c_+(1-\mathrm{e}^{\mathrm{i}kh})^\alpha + c_-(1-\mathrm{e}^{-\mathrm{i}kh})^\alpha + \mathrm{O}(h^{\alpha+1})] - \frac{b\tau}{h}(-\mathrm{i}kh + \mathrm{O}(h^2)). \end{aligned} \tag{26}$$

We note that $\tilde{p}(\mathrm{e}^{\mathrm{i}0h}) = 1$, whereas we can use the result for $\kappa < 0$ by complex conjugation of the $\kappa > 0$ case. Hence, we treat in detail the case $\kappa > 0$.

Since $k = |k|\mathtt{sign}(k)$,

$$\begin{aligned}(1-\mathrm{e}^{\mathrm{i}kh})^\alpha &= (-\mathrm{i}kh + \mathrm{O}(h^2))^\alpha = (-\mathrm{i}kh)^\alpha(1+\mathrm{O}(h))^\alpha = (-\mathrm{i}\mathtt{sign}(k))^\alpha |k|^\alpha h^\alpha (1+\mathrm{O}(h))^\alpha\\ &= \mathrm{e}^{-\mathrm{i}\mathtt{sign}(k)\alpha\pi/2}|k|^\alpha h^\alpha + \mathrm{O}(h^{\alpha+1})\end{aligned}$$

and

$$(1-\mathrm{e}^{-\mathrm{i}kh})^\alpha = \mathrm{e}^{\mathrm{i}\mathtt{sign}(k)\alpha\pi/2}|k|^\alpha h^\alpha + \mathrm{O}(h^{\alpha+1}).$$

Inserting these results into (26), we obtain

$$\begin{aligned}\tilde{p}(\mathrm{e}^{\mathrm{i}kh}) &= 1 - \frac{a\tau}{h^\alpha}[|k|^\alpha h^\alpha (c_+\mathrm{e}^{-\mathrm{i}\mathtt{sign}(k)\alpha\pi/2} + c_-\mathrm{e}^{\mathrm{i}\mathtt{sign}(k)\alpha\pi/2}) + \mathrm{O}(h^{\alpha+1})] + \mathrm{i}b\tau k + \tau\mathrm{O}(h)\\ &= 1 - a\tau|k|^\alpha(c_+\mathrm{e}^{-\mathrm{i}\mathtt{sign}(k)\alpha\pi/2} + c_-\mathrm{e}^{\mathrm{i}\mathtt{sign}(k)\alpha\pi/2}) + \mathrm{i}b\tau k + \tau\mathrm{O}(h).\end{aligned}$$

By use of (9) for c_- and c_+ and fixed $k > 0$, we have

$$\begin{aligned}c_+\mathrm{e}^{-\mathrm{i}\mathtt{sign}(k)\alpha\pi/2} + c_-\mathrm{e}^{\mathrm{i}\mathtt{sign}(k)\alpha\pi/2} &= \frac{\sin(\alpha-\theta)\pi/2}{\sin(\alpha\pi)}\left(\cos\frac{\alpha\pi}{2} - \mathrm{i}\mathtt{sign}(k)\sin\frac{\alpha\pi}{2}\right)\\ &\quad + \frac{\sin(\alpha+\theta)\pi/2}{\sin(\alpha\pi)}\left(\cos\frac{\alpha\pi}{2} + \mathrm{i}\mathtt{sign}(k)\sin\frac{\alpha\pi}{2}\right)\\ &= \frac{1}{\sin(\alpha\pi)}\left(\sin\frac{\alpha\pi}{2}\cos\frac{\theta\pi}{2} - \cos\frac{\alpha\pi}{2}\sin\frac{\theta\pi}{2}\right)\left(\cos\frac{\alpha\pi}{2} - \mathrm{i}\mathtt{sign}(k)\sin\frac{\alpha\pi}{2}\right)\\ &\quad + \frac{1}{\sin(\alpha\pi)}\left(\sin\frac{\alpha\pi}{2}\cos\frac{\theta\pi}{2} + \cos\frac{\alpha\pi}{2}\sin\frac{\theta\pi}{2}\right)\left(\cos\frac{\alpha\pi}{2} + \mathrm{i}\mathtt{sign}(k)\sin\frac{\alpha\pi}{2}\right)\\ &= \frac{1}{\sin(\alpha\pi)}\left(2\cos\frac{\theta\pi}{2}\right)\sin\frac{\alpha\pi}{2}\cos\frac{\alpha\pi}{2} + 2\mathrm{i}\mathtt{sign}(k)\sin\frac{\theta\pi}{2}\sin\frac{\alpha\pi}{2}\cos\frac{\alpha\pi}{2})\\ &= \cos\frac{\theta\pi}{2} + \mathrm{i}\mathtt{sign}(k)\sin\frac{\theta\pi}{2} = \mathrm{e}^{\mathrm{i}\mathtt{sign}(k)\theta\pi/2}.\end{aligned}$$

Thus,

$$\tilde{p}(\mathrm{e}^{\mathrm{i}kh}) = 1 - a\tau|k|^\alpha \mathrm{e}^{\mathrm{i}\mathtt{sign}(k)\theta\pi/2} + \mathrm{i}b\tau k + \tau\mathrm{O}(h).$$

Finally, by the definition of $\tilde{p}(z)$ in (23) and the relation of $\tilde{p}(\mathrm{e}^{\mathrm{i}kh})$ and $\tilde{y}(\mathrm{e}^{\mathrm{i}kh}, t)$ in (24), we obtain

$$\begin{aligned}\log(\tilde{y}(\mathrm{e}^{\mathrm{i}kh}, t)) &= \frac{t}{\tau}(\log(\tilde{p}(\mathrm{e}^{\mathrm{i}kh}))) = \frac{t}{\tau}\log(1 - a\tau|k|^\alpha\mathrm{e}^{\mathrm{i}\mathtt{sign}(k)\theta\pi/2} + \mathrm{i}b\tau k + \tau\mathrm{O}(h))\\ &= \frac{t}{\tau}(-a\tau|k|^\alpha\mathrm{e}^{\mathrm{i}\mathtt{sign}(k)\theta\pi/2} + \mathrm{i}b\tau k + \tau\mathrm{O}(h)) = -ta|k|^\alpha\mathrm{e}^{\mathrm{i}\mathtt{sign}(k)\theta\pi/2} + \mathrm{i}tbk + \mathrm{O}(h).\end{aligned}$$

Hence, (25) is obtained as desired. $\square$

4. An explicit finite difference scheme for LFADE in a bounded domain

In this section we consider LFADE in a bounded spatial domain $[0, R]$ with the following initial and boundary conditions:

$$\begin{array}{ll} \frac{\partial u(x,t)}{\partial t} = aD_\theta^\alpha u(x,t) - b\frac{\partial u(x,t)}{\partial x}, & 0 < x < R, \quad 0 < t < T, \\ u(x,0) = \varphi(x), & 0 < x < R, \\ u(0,t) = u(R,t) = 0, & 0 < t < T. \end{array} \tag{27}$$

We now discretize space and time by grid points and time instants as follows:

$$x_j = jh, j = 0,1,2,\ldots,N, h = \frac{R}{N}; t_n = n\tau, n = 0,1,2,\ldots,K, \tau = \frac{T}{K},$$

where h and τ are the space and time steps, respectively. Then, we can discretize the variable $u_j^n = u(x_j, t_n)$.

In the following we discretize Eq. (27), where we have adopted a first-order difference quotient in time (and in space) at level $t = t_n$ (and $x = x_j$) for approximating the first-order time (and space) derivative. To approximate the operator ${}_hI_\pm^\alpha$ by ${}_hI_\pm^{-\alpha}$, we adopt the Grünwald–Letnikov discretization of the fractional derivatives (12).

We can obtain an explicit finite difference scheme (EFDA) for LFADE with the initial and boundary conditions (27) as

$$\frac{u_j^{n+1} - u_j^n}{\tau} = -\frac{a}{h^\alpha}\left[c_+\sum_{k=0}^{j+1}(-1)^k\binom{\alpha}{k}u_{j+1-k}^n + c_-\sum_{k=0}^{N-j+1}(-1)^k\binom{\alpha}{k}u_{j-1+k}^n\right] - b\frac{u_j^n - u_{j-1}^n}{h}, \quad j = 1,2,\ldots,N-1. \tag{28}$$

Together with the boundary conditions $u_0^n = u_N^n = 0$, Eq. (27) results in a linear system of equations, whose coefficient matrix A has entries:

$$a_{ij} = \begin{cases} (-1)^{j-i}\frac{a\tau}{h^\alpha}c_-\binom{\alpha}{j-i+1}, & \text{when} \quad j \geqslant i+2, i = 1,2,\ldots,N-3, \\ -\frac{a\tau}{h^\alpha}\left(c_+ + c_-\binom{\alpha}{2}\right), & \text{when} \quad j = i+1, i = 1,2,\ldots,N-2, \\ 1 + \frac{a\tau}{h^\alpha}\binom{\alpha}{1}(c_+ + c_-) - \frac{b\tau}{h}, & \text{when} \quad j = i = 1,2,\cdots,N-1, \\ -\frac{a\tau}{h^\alpha}(c_+\binom{\alpha}{2} + c_-) + \frac{b\tau}{h} & \text{when} \quad j = i-1, i = 2,3,\ldots,N-1, \\ (-1)^{i-j}\frac{a\tau}{h^\alpha}c_+\binom{\alpha}{i-j+1}, & \text{when} \quad j \leqslant i-2, i = 3,4,\ldots,N-1. \end{cases} \tag{29}$$

The resulting linear system of equations can then be written in the following matrix form:

$$U^{n+1} = AU^n$$

where $U^n = (U_1^n, U_2^n, \ldots, U_{N-1}^n)^{\mathrm{T}}$.

5. Analyses of stability and convergence of EFDA

In the above section, EFDA for LFADE has been presented. In this section we will discuss the stability and convergence of EFDA in a bounded domain. The stability of EFDA can be proved under the scaling restriction condition (21) of the discrete random walk model.

Theorem 2. *Under the assumption* (21), *EFDA* (28) *for LFADE is stable when* $1 < \alpha \leqslant 2$ *in a bounded domain.*

Proof. Under the assumption (21), the transition coefficients (20) fulfil the conditions (18). We have that the sum of all elements in every row of the coefficient matrix A is less than the total sum of the transition coefficients, i.e., is less than 1. Thus, we obtain

$||A||_\infty < 1.$

According to the Lax–Richtmer definition of stability [25], we obtain that EFDA (28) for LFADE is stable when $1 < \alpha \leqslant 2$ in a bounded domain under the condition (21). □

To analyze the convergence, we find it worthwhile to recall here the following useful lemma associated with the error estimate proposition referred to in [12].

Lemma 1. *Suppose that $f \in L_1(\mathbb{R})$ and $f \in \ell^{\alpha+1}(\mathbb{R})$, and let*

$$_hI_+^{-\alpha} f(x) = \frac{1}{h^\alpha} \sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} f(x - (k-p)h),$$

where p is a nonnegative integer, $I_+^{-\alpha} f(x)$ is the left hand sided Riemann–Liouville (i.e., Grünwald–Letnikov) fractional derivative at interval $(-\infty, x)$. Then

$$_hI_+^{-\alpha} f(x) = I_+^{-\alpha} f(x) + \mathrm{O}(h)$$

uniformly in $x \in \mathbb{R}$ as $h \to 0$.

With respect to the right hand sided Riemann–Liouville (i.e., Grünwald–Letnikov) fractional derivative $I_-^{-\alpha}$ defined on the interval $(x, +\infty)$, we can establish a similar proposition to the left hand sided Riemann–Liouville fractional derivative:

$$_hI_-^{-\alpha} f(x) = I_-^{-\alpha} f(x) + \mathrm{O}(h)$$

uniformly in $x \in \mathbb{R}$ as $h \to 0$.

Theorem 3. *Let U be the exact solution of Eq. (27) and u be the numerical solution of the finite difference Eq. (28). Then u converges to U as h and τ tend to zero when the condition (21) is satisfied.*

Proof. Let error $e = U - u$, and at the mesh points $(x_j, t_n), u_j^n = U_j^n - e_j^n$. Substitution into the difference Eq. (28) leads to

$$\begin{aligned}\frac{(U_j^{n+1} - e_j^{n+1}) - (U_j^n - e_j^n)}{\tau} &= -\frac{a}{h^\alpha}\left[c_+ \sum_{k=0}^{j+1} (-1)^k \binom{\alpha}{k} (U_{j+1-k}^n - e_{j+1-k}^n)\right] \\ &\quad - \frac{a}{h^\alpha}\left[c_- \sum_{k=0}^{N-j+1} (-1)^k \binom{\alpha}{k} (U_{j-1+k}^n - e_{j-1+k}^n)\right] - b\frac{(U_j^n - e_j^n) - (U_{j-1}^n - e_{j-1}^n)}{h},\end{aligned}$$

i.e.,

$$\begin{aligned}\frac{(U_j^{n+1} - U_j^n) - (e_j^{n+1} - e_j^n)}{\tau} &= -\frac{a}{h^\alpha}\left[c_+ \sum_{k=0}^{j+1} (-1)^k \binom{\alpha}{k} U_{j+1-k}^n + c_- \sum_{k=0}^{N-j+1} (-1)^k \binom{\alpha}{k} U_{j-1+k}^n\right] \\ &\quad - \frac{a}{h^\alpha}\left[c_+ \sum_{k=0}^{j+1} (-1)^k \binom{\alpha}{k} e_{j+1-k}^n + c_- \sum_{k=0}^{N-j+1} (-1)^k \binom{\alpha}{k} e_{j-1+k}^n\right] \\ &\quad - b\frac{(U_j^n - U_{j-1}^n) - (e_j^n - e_{j-1}^n)}{h}.\end{aligned} \tag{30}$$

According to the operators $_hI_\pm^{-\alpha}$ in (15) and $_hD_\theta^\alpha$ in (14), the first term on the right-side of Eq. (30) can be written as

$$-\frac{a}{h^\alpha}\left[c_+ \sum_{k=0}^{j+1} (-1)^k \binom{\alpha}{k} U_{j+1-k}^n + c_- \sum_{k=0}^{N-j+1} (-1)^k \binom{\alpha}{k} U_{j-1+k}^n\right] = -a[c_{+h}I_+^{-\alpha} U_j^n + c_{-h}I_-^{-\alpha} U_j^n] = a[_hD_\theta^\alpha U]_j^n.$$

From Lemma 1, we have

$$D_\theta^\alpha = -[c_+ I_+^{-\alpha} + c_- I_-^{-\alpha}] = -[c_+({}_hI_+^{-\alpha} + \mathrm{O}(h)) + c_-({}_hI_-^{-\alpha} + \mathrm{O}(h))]$$
$$= -[c_+ {}_hI_+^{-\alpha} + c_- {}_hI_-^{-\alpha}] + \mathrm{O}(h) = {}_hD_\theta^\alpha + \mathrm{O}(h).$$

Using Taylor's theorem, we have

$$\frac{U_j^{n+1} - U_j^n}{\tau} = \left(\frac{\partial U}{\partial t}\right)_j^n + \mathrm{O}(\tau)$$

and

$$\frac{U_j^n - U_{j-1}^n}{h} = \left(\frac{\partial U}{\partial x}\right)_j^n + \mathrm{O}(h).$$

Consequently, we obtain

$$\frac{e_j^{n+1} - e_j^n}{\tau} = -\frac{a}{h^\alpha}\left[c_+ \sum_{k=0}^{j+1} (-1)^k \binom{\alpha}{k} e_{j+1-k}^n + c_- \sum_{k=0}^{N-j+1} (-1)^k \binom{\alpha}{k} e_{j-1+k}^n\right] - b\frac{e_j^n - e_{j-1}^n}{h} + \mathrm{O}(\tau + h).$$

Using the initial and boundary conditions $e_j^0 = 0, j = 0, 1, \ldots, N$ and $e_0^n = e_N^n = 0, n = 0, 1, \ldots, K$, the above equation can be rewritten in matrix form as

$$R_{n+1} = AR_n + M, \quad R_0 = 0,$$

where $R_n = (e_1^n, e_2^n, \ldots, e_{K-1}^n)^{\mathrm{T}}, M = \tau\mathrm{O}(\tau + h)(1, 1, \ldots, 1)^{\mathrm{T}}$ and A is defined in (29). Hence, we can obtain

$$R_{n+1} = (A^n + A^{n-1} + \cdots + A + I)M.$$

Thus

$$\|R_{n+1}\|_\infty \leqslant (\|A^n\|_\infty + \|A^{n-1}\|_\infty + \cdots + \|A\|_\infty + \|I\|_\infty)\|M\|_\infty.$$

Since under the condition (21)

$$\|A\|_\infty < 1,$$

we obtain

$$\|R_{n+1}\|_\infty < (n + 1)\tau|\mathrm{O}(\tau + h)|.$$

Consequently, when $\tau \to 0$, $h \to 0$, we have $\|R_{n+1}\| \to 0$, i.e., $|e_{j+1}^n| \to 0$. This proves that u converges to U as τ and h tend to zero under the condition (21). □

6. Numerical examples

In this section, the following LFADE is considered:

$$\frac{\partial u(x,t)}{\partial t} = aD_\theta^\alpha u(x,t) - b\frac{\partial u(x,t)}{\partial x}, \quad 0 < x < \pi, \; 0 < t < T, \; 1 < \alpha \leqslant 2,$$
$$u(x,0) = \varphi(x) = \sin(x), \quad 0 \leqslant x \leqslant \pi,$$
$$u(0,t) = u(\pi,t) = 0, \quad 0 < t < T.$$

In order to demonstrate the efficiency of the EFDA, we first validate it through comparison of the numerical solution (EFDA) and the solution obtained by the latter method. This fractional method of lines (FMoL) was first introduced by Liu et al. [5–7] to solve fractional partial differential equations. The FMoL for LFADE can be written as

$$\frac{\mathrm{d}u_l}{\mathrm{d}t} = -\frac{a}{h^\alpha}\left(c_+ \sum_{k=0}^{l+1} (-1)^k \binom{\alpha}{k} u_{l+1-k} + c_- \sum_{k=0}^{N-l+1} (-1)^k \binom{\alpha}{k} u_{l-1+k}\right) - b\frac{u_l - u_{l-1}}{h},$$
$$1 < \alpha \leqslant 2, \quad l = 1, 2, \ldots, K - 1,$$

where $u_j = u(x_i,t)$.

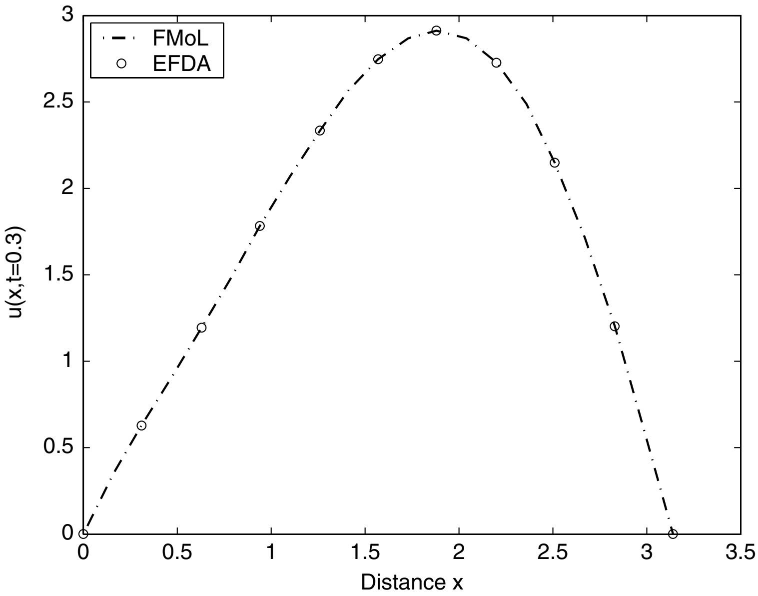

Fig. 1. The numerical solutions (FMoL) and EFDA for $\alpha = 1.7$, $\theta = 0.3$, $a = 1.5$, $b = 1.0$, $t = 0.3$.

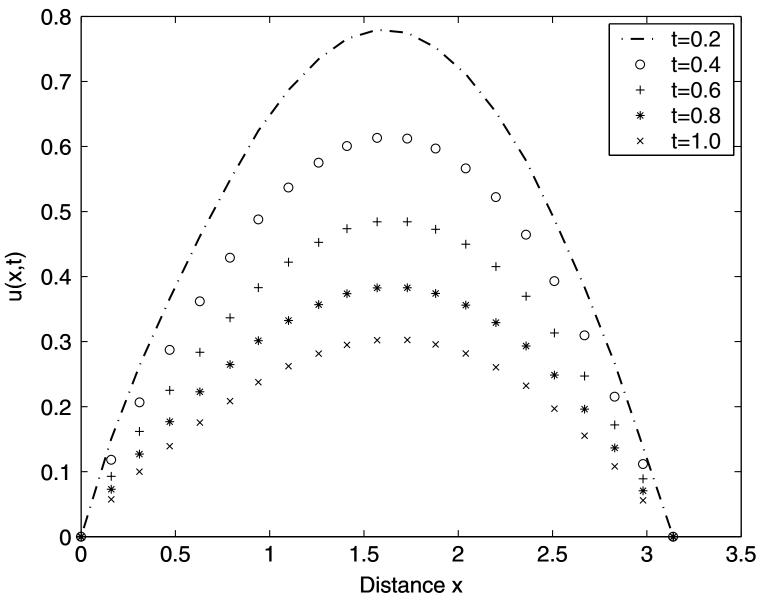

Fig. 2. EFDA for $\alpha = 1.7$, $\theta = 0.3$, $a = 1.5$, $b = 1.0$, $t \in (0,1)$.

In Fig. 1, the numerical solutions (FMoL) and EFDA for $\alpha = 1.7$, $\theta = 0.3$, $a = 1.5$, $b = 1.0$ are shown. It is apparent from the figure that EFDA is in good agreement with the numerical solution. Fig. 2 shows the evolution results using EFDA with $h = \pi/100$, $\tau = 0.0001$, $\alpha = 1.7$, $\theta = 0.3$, $a = 1.5$, $b = 1.0$ ($0 \leqslant t \leqslant 1, 0 \leqslant x \leqslant \pi$).

Fig. 3 shows the response of the advection–dispersion process using EFDA for different θ, which indicates the skewness.

Fig. 4 shows the response of the advection–dispersion process using EFDA for different diffusion coefficients a. It indicates that the solution decays more quickly while the diffusion coefficient a increases.

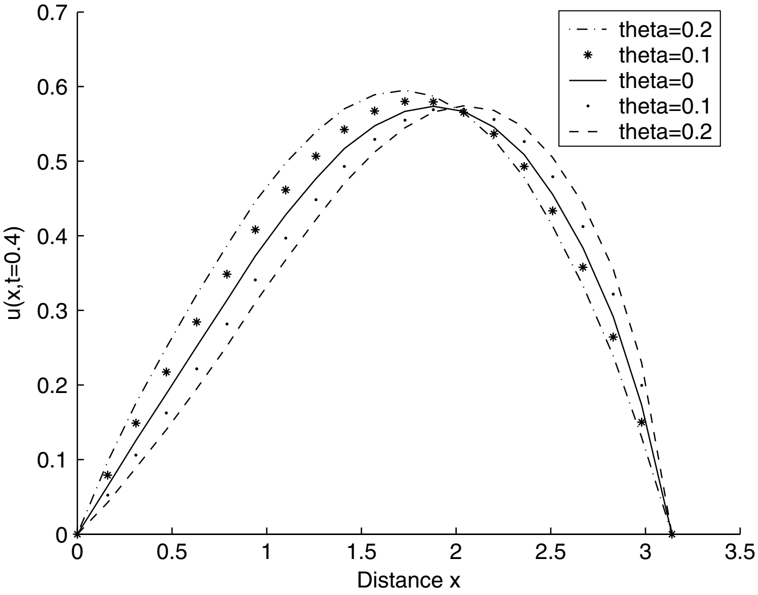

Fig. 3. EFDA for $\alpha = 1.7$, $a = 1.5$, $b = 1.0$, $\theta = \pm 0.2$, ± 0.1, 0, $t = 0.4$.

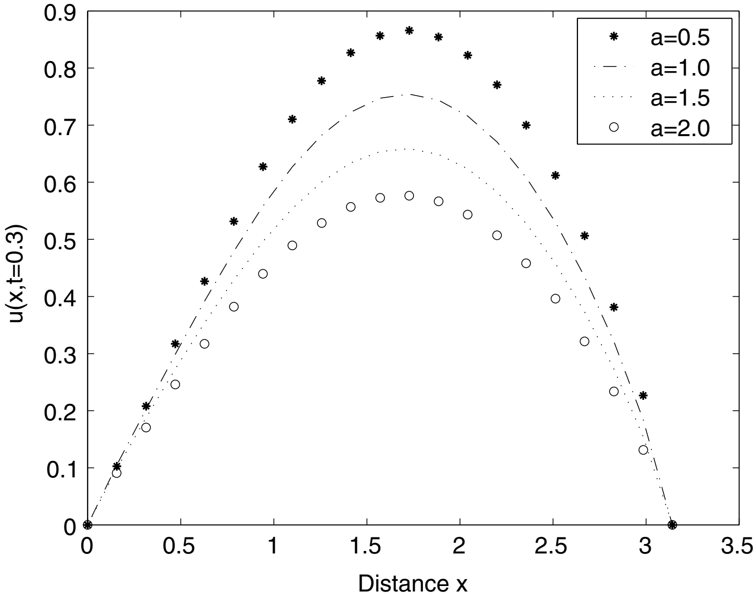

Fig. 4. EFDA for $\alpha = 1.7$, $b = 1.0$, $\theta = 0.3$, $a = 0.5$, 1.0, 1.5, 2.0, $t = 0.3$.

In order to demonstrate again the efficiency of the EFDA, we take the parameters $\alpha = 2$, $\theta = 0$, $b = 0$ and the initial and boundary conditions

$$\begin{aligned} &u(x,0) = \varphi(x) = x^2(\pi - x), \quad 0 \leqslant x \leqslant \pi, \\ &u(0,t) = u(\pi,t) = 0, \quad t \geqslant 0. \end{aligned} \tag{31}$$

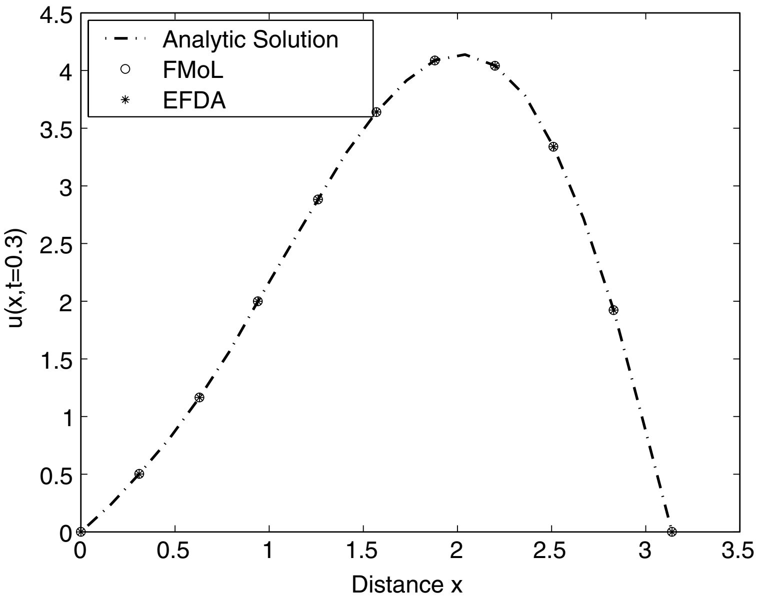

Fig. 5. The analytical solution, numerical solutions (FMoL) and EFDA for $\alpha = 2$, $\theta = 0$, $a = 0.25$, $b = 0$, $t = 0.3$.

Table 1
Comparison of EFDA and FMoL for $a = 1.5$, $b = 0$, $\alpha = 1.7$, $\theta = 0.3$, $t = 0.3$, $h = \pi/100$ and different τ

x_i	$\tau = 0.001$ (EFDA)	$\tau = 0.00115$ (EFDA)	(FMoL)
0.3142	0.23041473	230.54720622	0.23017721
0.6283	0.40603728	577.16038497	0.40562519
0.9425	0.54876256	869.39062934	0.54821574
1.2566	0.64661685	792.32079213	0.64598244
1.5708	0.68848748	437.27120064	0.68781812
1.8850	0.66770824	145.44209209	0.66706004
2.1991	0.58292127	28.89777087	0.58235311
2.5133	0.43764813	3.54256895	0.43721936
2.8274	0.23952071	0.40074543	0.23928533

The analytical solution [26] is

$$u(x,t) = \sum_{k=1}^{\infty} \left(\frac{8(-1)^{k+1} - 4}{k^3} \right) \sin(kx) \mathrm{e}^{-ak^2 t}.$$

In Fig. 5, the numerical solutions FMoL, EFDA and the analytical solution of LFADE with initial and boundary conditions (31) are shown for a special case $\alpha = 2$, $\theta = 0$, $a = 0.25$, $b = 0.0$. From Fig. 5, it can be seen that our computed result is in good agreement with both FMoL and the analytical solution.

In Figs. 1–5 h and τ satisfy the scaling restriction condition (22), then (21).

In order to examine the scaling restriction condition (21), a comparison of the numerical solutions for both EFDA and FMoL is listed in Table 1 for the case with $a = 1.5$, $b = 0$, $\alpha = 1.7$, $\theta = 0.3$, $t = 0.3$.

From Table 1 it can be seen that when the restriction condition (21) of stability is fulfilled, the results gained from EFDA is close to the results gained from FMoL. However, when the restriction condition (21) is not fulfilled, the results from EFDA do not match those from FMoL, which are in good agreement with the theory.

7. Conclusions

In this paper we have generated a discrete random walk model for LFADE. Under the restriction condition (21), we also prove that the discrete random walk model converges to the related LFADE. Then EFDA for

LFADE is presented, and the stability and convergence of the EFDA are discussed. Finally, some numerical results are presented to show the application of the present technique, and rigorous analysis of the theory is demonstrated.

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